

NON-TRIVIAL SELF-CONCORDANCES AND A RECENT CONJECTURE BY BOTVINNIK

WOLFGANG STEIMLE

ABSTRACT. The goal of this note is to construct, on many manifolds, non-trivial concordances from the identity to itself. This produces counterexamples to a recent conjecture by Botvinnik.

1. STATEMENT OF THE RESULTS

Recall that a (smooth) concordance on a smooth manifold M is a diffeomorphism of $M \times I$ which is the identity in a neighborhood of $M \times 0 \cup \partial M \times I$. For a concordance H of M , denote by $e(H)$ the induced diffeomorphism on $M \times 1$. We say that H is *trivial* if it is isotopic to the identity, via an isotopy that fixes a neighborhood of $M \times 0 \cup \partial M \times I$.

Concordances can be described, in a stable range, by algebraic K -theory. In this note we explain how to use this relationship to prove:

Theorem 1.1. *For $n \geq 9$ there exists a non-trivial concordance H of $S^1 \times D^{n-1}$ such that $e(H) = \text{id}$.*

In fact, more generally we have:

Theorem 1.2. *On any smooth compact orientable manifold M of dimension $n \geq 9$ such that $\pi_1(M) = \mathbb{Z}$, there is a non-trivial concordance H such that $e(H) = \text{id}$.*

Recently, in his impressive preprint [1], Botvinnik has proposed the following “Topological Conjecture”.

Conjecture 1.3. *Let M be a closed manifold equipped with a metric g of positive scalar curvature. If H is a non-trivial concordance on M , then g and $e(H)^*g$ are non-isotopic as metrics of positive scalar curvature.*

Corollary 1.4. *The Topological Conjecture does not hold.*

2. PROOF OF THEOREM 1.1

Write $M = S^1 \times D^{n-1}$ and denote by $C(M)$ the group of concordances modulo isotopy. Let $h \in C(M)$. Shrinking the interval $I = [0, 1]$ to $[0, \frac{1}{2}]$, we may consider h as a self-diffeomorphism of $M \times [0, \frac{1}{2}]$ which we may extend again to the whole of $M \times I$ by “flipping”:

$$H|_{M \times [\frac{1}{2}, 1]} := i \circ h \circ i$$

where i is induced by reflection of I at $\frac{1}{2}$. Clearly $e(H) = \text{id}$.

Remark 2.1. A further analysis using the Hatcher spectral sequence and surgery theory shows that on the manifold M any concordance from the identity to itself is isotopic to one of this form.

Note that in the abelian group $C(M)$, we have $H = h + \tau h$ where τ denotes the canonical involution on $C(M)$. To prove Theorem 1.1 we therefore need to show that $\tau \neq -\text{id}$.

By the stable parametrized h -cobordism theorem [8] there is a short exact sequence

$$0 \rightarrow \pi_2^s(M \amalg \{*\}) \rightarrow \pi_2 A(M) \xrightarrow{\pi} C(M) \rightarrow 0$$

provided $\dim(M) \geq 9$ (this “stable range” is due to [5]). Here $A(M)$ denotes Waldhausen’s K -theory of spaces [7]; by homotopy invariance we have $\pi_2 A(M) \cong \pi_2 A(S^1)$. By [6] the functor $A(-)$ carries a canonical involution T so that the map π is equivariant up to the sign $(-1)^n$. (Here we use that M is parallelizable.)

By the fundamental theorem [3]

$$(1) \quad \pi_2 A(S^1) \cong \pi_2 A(*) \oplus \pi_1 A(*) \oplus \pi_2 N A(*) \oplus \pi_2 N A(*);$$

the involution T interchanges the two copies of $\pi_2 N A(*)$ by [4]. Moreover $\pi_k A(*) \cong \pi_k^s$ for $k \leq 2$ in a way that

$$\pi_2 A(*) \oplus \pi_1 A(*) = \pi_2^s(S^1 \amalg \{*\})$$

as subgroups of $\pi_2 A(S^1)$. So

$$C(M) \cong \pi_2 N A(*) \oplus \pi_2 N A(*)$$

and τ acts, up to sign, by interchanging the summands. In particular $\tau \neq -\text{id}$ (with $\pi_2 C(M)$ being non-zero by [2]).

3. PROOF OF THEOREM 1.2

We may assume that M is connected. Let H be a non-trivial concordance on $S^1 \times D^{n-1}$ as given by Theorem 1.1. Let $i: S^1 \rightarrow M$ be an embedding representing the generator of $\pi_1(M)$. Since M is oriented, i has a trivial normal bundle and induces an embedding $\bar{i}: S^1 \times D^{n-1} \rightarrow M$. So we may extend H by the identity to a concordance \bar{H} on M , such that $e(\bar{H}) = \text{id}$.

In the stable range $\dim(M) \geq 9$, the assignment $M \mapsto C(M)$ is a homotopy functor [2]. Thus, if $\rho: M \rightarrow S^1 \times D^{n-1}$ classifies (up to homotopy) the universal covering of M , then the composite homomorphism

$$C(S^1 \times D^{n-1}) \xrightarrow{\bar{i}_*} C(M) \xrightarrow{\rho_*} C(S^1 \times D^{n-1})$$

is the identity. Hence $\bar{H} = \bar{i}_*(H) \neq 0$.

REFERENCES

- [1] B. Botvinnik. Concordance and isotopy of metrics with positive scalar curvature. 2012.
- [2] A. E. Hatcher. Concordance spaces, higher simple-homotopy theory, and applications. In *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 1*, pages 3–21. Amer. Math. Soc., Providence, R.I., 1978.
- [3] T. Hüttemann, J. R. Klein, W. Vogell, F. Waldhausen, and B. Williams. The “fundamental theorem” for the algebraic K -theory of spaces. I. *J. Pure Appl. Algebra*, 160(1):21–52, 2001.
- [4] T. Hüttemann, J. R. Klein, W. Vogell, F. Waldhausen, and B. Williams. The “fundamental theorem” for the algebraic K -theory of spaces. II. The canonical involution. *J. Pure Appl. Algebra*, 167(1):53–82, 2002.
- [5] K. Igusa. The stability theorem for smooth pseudoisotopies. *K-Theory*, 2(1-2):vi+355, 1988.
- [6] W. Vogell. The involution in the algebraic K -theory of spaces. In *Algebraic and geometric topology (New Brunswick, N.J., 1983)*, volume 1126 of *Lecture Notes in Math.*, pages 277–317. Springer, Berlin, 1985.
- [7] F. Waldhausen. Algebraic K -theory of spaces. In *Algebraic and geometric topology (New Brunswick, N.J., 1983)*, pages 318–419. Springer-Verlag, Berlin, 1985.
- [8] F. Waldhausen, B. Jahren, and J. Rognes. Spaces of PL manifolds and categories of simple maps. Preprint, 2008.

UNIVERSITÄT BONN, MATHEMATISCHES INSTITUT, ENDENICHER ALLEE 60, D-53115 BONN,
GERMANY

E-mail address: `steimle@math.uni-bonn.de`